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# Small-world phenomena in physics: the Ising model

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**Abstract.** The Ising system with a small fraction of random long-range interactions is the simplest example of small-world phenomena in physics. Considering the latter both in an annealed and in a quenched state we conclude that: (a) the existence of random long-range interactions leads to a phase transition in the one-dimensional case and (b) there is a minimal average number  $p$  of these interactions per site ( $p < 1$  in the annealed state, and  $p \simeq 1$  in the quenched state) needed for the appearance of the phase transition. Note that the average number of these bonds,  $pN/2$ , is much smaller than the total number of bonds,  $N^2/2$ .

## 1. Introduction

The remarkable paper of Watts and Strogatz on small-world networks [1] has drawn the attention of many researchers in different fields [2]. Although the idea of the crucial importance of random long-range connections for the self-organization of a large networks by itself is not new (one needs ‘six degree of separation’ to connect two randomly chosen people in the USA [3]), Watts and Strogatz have formulated the quantitative theory of this phenomenon which has many applications in science including physics. The Ising model can be used to exemplify the simplest example of the small-world phenomena in physics. The influence of random long-range interactions on the Ising system with nearest-neighbour interaction is quite different in the one- and two–three-dimensional cases.

Although two–three-dimensional Ising models are obviously more relevant to physical systems, even the one-dimensional Ising model with random long-range interactions can be applied to the description of systems such as magnetic linear polymers where the short-range interactions between neighbouring monomers in the chain are supplemented by the random interactions between monomers that are close in space (and not along the chain) [4]. Nevertheless, in this case the fraction of these long-range interactions tends to zero in the thermodynamic limit [5], and there is no phase transition at non-zero temperature [6].

In the subsequent discussion, we consider the one-dimensional Ising model with a short-range interaction  $J$  and random long-range interactions  $I_{ij}$  acting between a small number  $pN/4$  pairs  $(i, j)$  of spins, where  $k$  is the coordination number ( $k = 2$  for the one-dimensional Ising model) and  $p$  is the average number of long-range interaction per site.

Hence, the Hamiltonian  $H$  of  $N$  spins in the external magnetic field  $B$  with periodic boundary conditions has the form

$$H = - \sum_{i=1}^N J \sigma_i \sigma_{i+1} - \sum_{i>j}^N I_{ij} \sigma_i \sigma_j - B \sum_{i=1}^N \sigma_i. \quad (1)$$

The probability distribution for the  $I_{ij}$  is given by

$$P(I_{ij}) = \frac{p}{N} \delta(I_{ij} - I) + \left(1 - \frac{p}{N}\right) \delta(I_{ij}). \quad (2)$$

For definiteness, we assume that all interactions are ferromagnetic,  $J, I > 0$ . For  $p = 0$  (ordered system), no phase transition takes place in the one-dimensional case, while such a transition (with the so-called Ising critical indices) exists in two–three dimensions. On the other hand, for  $p = N$  (fully disordered system), the phase transition (with the so-called mean-field critical indices) exists both in one and in two–three dimensions.

The Erdos–Renyi theorem for the random graphs [7] (which are slightly different from the small-world networks [8]) establishes that long-range order appears at  $p \simeq 1$  (and not  $p \simeq N$ ). Hence, one expects that the same phenomenon will occur for small-world networks. Indeed, in a recent paper [8], Barrat and Weigt have shown that in the absence of the short-range interaction ( $J = 0$  in (1)) long-range order appears at a temperature  $T \approx -\frac{k(k+2)}{4 \lg(p)}$  for small  $p$ . In other words, in order to obtain a phase transition at non-zero temperatures in the one-dimensional Ising system, there is no need for a long-range Kac potential acting among all spins—it is sufficient to have a small number of random long-range interactions, and they will ensure (small-world effect!) the appearance of long-range order.

Studies of the influence of the long-range interactions on the Ising systems have a long history that goes back to the 1960–70s [9, 10]. At the same time, some interest was shown in the dilute ferromagnet which was expressed by consideration of random short-range interactions [12, 13]. Finally, both long-range interactions and random short-range interactions are a subject of study in spin-glass theory [14]. Our method of analysis of the Hamiltonian (1) is similar to those mentioned above, although we are interested in the slightly different case of random long-range interactions.

The random systems described by the Hamiltonian (1) can have two different types of thermodynamic behaviour [15], the annealed case when the interactions between spins are able to reach thermal equilibrium at each temperature, and the quenched case when the interactions are frozen. In line with this, one has to perform an average over the random interactions in the statistical sum  $Z = -\exp(-H/\kappa T)$  for the annealed case, or in the free energy  $F = -\kappa T \ln Z$  for the quenched case. In fact, the annealed free energy presents a lower bound for the free energy for the following reason. In this case the bonds can arrange themselves to minimize the free energy, while in the quenched case the bonds are frozen and, therefore, are unable to choose the energetically most favourable positions.

We start with the simpler although physically less common annealed case. The exact solution is given in section 2. Section 3 contains an approximate solution of the annealed case which is used in section 4 for the analysis of the quenched case. Finally, some conclusions are presented in section 5.

## 2. Annealed case: exact solution

For the annealed case, one can use a thermodynamic rather than the statistical-mechanical approach which was used [12, 13] for the analysis of dilute ferromagnets. Let us write in the following form the Hamiltonian  $H$  of  $N$  spins in an external magnetic field  $B$ :

$$H = -J \sum_{i=1}^N \sigma_i \sigma_{i+1} - I \sum_{i,j}^N \mu_{ij} \sigma_i \sigma_j - B \sum_{i=1}^N \sigma_i \quad (3)$$

where the random variable  $\mu_{ij}$  denotes the presence ( $\mu_{ij} = 1$ ) or the absence ( $\mu_{ij} = 0$ ) of a long-range interaction between sites  $i$  and  $j$ , while the fraction of these interactions is equal

to  $p$ ,

$$\frac{\sum_{bonds} \mu_{ij}}{N} = p. \quad (4)$$

The thermodynamic behaviour of the system is defined by the grand canonical partition function with pseudo-chemical potential  $\xi$ ,

$$\Xi(I, J, B, \xi) = \sum_{p=0}^N \exp(\beta p \xi) \text{Tr}_{\sigma} \sum_{\mu_{ij}=0,1} \exp \left[ \beta I \sum_{bonds} \mu_{ij} \sigma_i \sigma_j + \sum_{i=1}^N \beta (J \sigma_i \sigma_{i+1} + B \sigma_i) \right] \quad (5)$$

where  $\beta = 1/\kappa T$ .

The summation over  $\mu_{ij}$  subject to condition (4) can be performed at once,

$$\sum_{\mu_{ij}=0,1} \exp [\mu_{ij} (\beta I \sigma_i \sigma_j + \beta \xi)] = 1 + \exp (\beta I \sigma_i \sigma_j + \beta \xi) \equiv A \exp (\beta K \sigma_i \sigma_j). \quad (6)$$

The last equality in (6) allows us, with  $A$  and  $K$  suitably chosen functions of  $I$  and  $\xi$ , to reduce our problem to the ordinary Ising problem with short- and long-range interactions (the so-called ‘reference system’). Equation (6) is satisfied for  $\sigma_i \sigma_j$  equal to  $\pm 1$  if

$$\exp(\beta \xi) = \frac{\sinh(\beta K)}{\sinh[\beta(I - K)]} \quad A = \frac{\sinh(\beta I)}{\sinh[\beta(I - K)]}. \quad (7)$$

Substituting (6) into (5), one finds

$$\Xi(B, T|I, J, \xi) = A^N \text{Tr}_{\sigma} \exp \left[ \beta K \sum_{i,j} \sigma_i \sigma_j + \sum_{i=1}^N \beta (J \sigma_i \sigma_{i+1} + B \sigma_i) \right] \quad (8)$$

where the summation  $i, j$  extends over all bonds.

Due to the presence of long-range interactions in the Hamiltonian (3), the thermodynamic behaviour described by the statistical sum (8) or by the free energy  $F = -\frac{1}{\beta} \ln \Xi$ , comes under the heading of the mean field, and can be investigated by means of the Landau expansion of the thermodynamic potential written in the appropriate variables [10].

Let us start with the case when only short-range, nearest-neighbour (nn) interactions are present ( $K = 0$ ). Then, the free energy  $F_{nn}$  can be calculated by the transfer-matrix technique, which gives

$$\beta F_{nn}(B, T) = -\beta J - \ln [\cosh(\beta B) + \sqrt{\sinh^2(\beta B) + \exp(-4\beta J)}] \quad (9)$$

from which the magnetic moment  $M$  conjugated to the magnetic field  $B$  is equal to [11]

$$M = \left( \frac{\partial F_{nn}}{\partial B} \right)_{\beta} = \frac{\sinh(\beta B)}{\sqrt{\sinh^2(\beta B) + \exp(-4\beta J)}}. \quad (10)$$

Performing the Legendre transformation of the basic thermodynamic variables  $M, T$ , one obtains

$$F_{nn}(M, T) = F_{nn}(B, T) + MB. \quad (11)$$

The classical way of taking into account the long-range interactions consists of calculating their contribution to the energy of a state when one has  $n$  ‘plus’ spins and  $N - n$  ‘minus’ spins. Then, the energy will be

$$-K [n^2 + (N - n)^2 - 2n(N - n)] = -K(N - 2n)^2 = -KM^2. \quad (12)$$

Combining (11) and (12), one obtains

$$F(M, T) = F_{nn}(B, T) + MB - KM^2. \quad (13)$$

The critical temperature can be found from (13) by two equivalent methods. Firstly, near the critical point one can expand the free energy (13) in a series in  $M$  in the absence of the external field, or, alternatively, one can obtain from (13) the following equation for the order parameter:

$$M = \frac{\sinh(\beta B + 2\beta K M)}{\sqrt{\sinh(\beta B + 2\beta K M) + \exp(-4\beta J)}} \quad (14)$$

which is a natural generalization of (10).

The equation for the critical point can now be obtained by the appearance of the non-trivial solution,  $M \neq 0$ , in (14) in the absence of an external field which gives the following condition for the critical point:

$$\frac{2K}{T_c} \exp\left(\frac{2J}{\kappa T_c}\right) = 1. \quad (15)$$

The long-range interaction of the reference system introduced in (8) can be expressed in terms of the strength  $I$  and the concentration  $p$  of the long-range interactions in the original system. The concentration of the long-range interactions  $p$  is equal to the average number,  $\langle n \rangle$ , of the existing long-range bonds, namely

$$p = \langle n \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \frac{\partial \ln \Xi}{\partial \xi} = \frac{\partial \ln A}{\partial \xi} + \varepsilon \frac{\partial(\beta K)}{\partial \xi} \quad (16)$$

where

$$\varepsilon \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \frac{\partial \ln [\exp \sum_{i,j} (\beta K \sigma_i \sigma_j)]}{\partial(\beta K)} \right\rangle = \langle \sigma_i \sigma_j \rangle \quad (17)$$

is the 2-spin correlation function of the reference system.

Using (7), one can rewrite (16) in the following form:

$$\exp\left(\frac{2I}{\kappa T}\right) = \frac{p + p_2}{p - p_1} \quad (18)$$

where

$$p_1 = \frac{1}{2} \left[ 1 - \exp\left(-\frac{2K}{\kappa T}\right) \right] (1 + \varepsilon) \quad p_2 = \frac{1}{2} \left[ \exp\left(\frac{2K}{\kappa T}\right) - 1 \right] (1 - \varepsilon). \quad (19)$$

The reference system in zero external field has a mean-field phase transition at some  $T = T_c$ . Therefore, the original system described by the Hamiltonian (3) will have a phase transition, if any, at  $T = T_c$  which satisfies (18). The critical temperature  $T_c$  can be expressed in terms of the original parameters  $I$  and  $J$  by excluding the auxiliary parameter  $K$  from (15) and (18). However, a few important conclusions can already be drawn from (18). Both  $p_1$  and  $p_2$  are positive, and  $p_1 < 1$ . Therefore, a phase transition at a non-zero temperature occurs only for  $p_1 < p < 1$ , so that  $p_1$  represents a minimum fraction of the random long-range interactions below which no phase transition is possible.

### 3. Annealed case: approximate solution

Returning to the Hamiltonian (1) and performing the average over  $I_{ij}$  given by the distribution (2), one obtains for the canonical statistical sum

$$\begin{aligned}\langle Z \rangle_I &= \text{Tr}_\sigma \int dI_{ij} P(I_{ij}) \exp(-\beta H) \\ &= \text{Tr}_\sigma \exp \left( \sum_{i=1}^N [\beta J \sigma_i \sigma_{i+1} + \beta \mu_0 B \sigma_i] \right) \prod_{i,j} \left[ 1 - \frac{p}{N} + \frac{p}{N} \exp(\beta I \sigma_i \sigma_j) \right] \\ &= \text{Tr}_\sigma \exp \left[ \sum_{i=1}^N [\beta J \sigma_i \sigma_{i+1} + \beta \mu_0 B \sigma_i] + \sum_{i,j} \ln \left( 1 + \frac{p}{N} [\exp(\beta I \sigma_i \sigma_j) - 1] \right) \right]\end{aligned}\quad (20)$$

where the obvious identity  $\prod_{i,j} \dots = \exp \sum_{i,j} \ln \dots$  has been used in the last part of equation (20).

One can rewrite the last exponent in (20) in the following form:

$$\exp(\beta I) \sigma_i \sigma_j = \cosh(\beta I) [1 + \sigma_i \sigma_j \tanh(\beta I)]. \quad (21)$$

Substituting (21) into (20) and expanding the logarithm in a series in  $p/N$ , one finds

$$\langle Z \rangle_I = \text{Tr}_\sigma \exp \left[ \sum_i [\beta J \sigma_i \sigma_{i+1} + \beta \mu_0 H \sigma_i] + \frac{p}{N} (\cosh \beta I - 1) + \frac{p \sinh \beta I}{N} \sum_{i,j} \sigma_i \sigma_j \right] \quad (22)$$

where only terms linear in  $p/N$  were retained in the expansion of the logarithm in the series since only these terms contribute extensively to the free energy.

The final step in the calculation will be the transformation of the sum over  $i, j$  in (22) in the form  $\sum_{i,j} \sigma_i \sigma_j = \frac{1}{2} (\sum_i \sigma_i)^2 - N/2$ , and the use of the relation

$$\exp \left[ \frac{p \sinh \beta I}{2N} \left( \sum_i \sigma_i \right)^2 \right] = \frac{\sqrt{2}}{\sqrt{N\pi}} \int_{-\infty}^{\infty} d\phi \exp \left[ -\frac{\phi^2 N}{2} + \sqrt{p \sinh \beta I} \sum_i \sigma_i \phi \right] \quad (23)$$

which gives

$$\begin{aligned}\langle Z \rangle_I &= \frac{\sqrt{2}}{\sqrt{N\pi}} \int_{-\infty}^{\infty} d\phi \exp(-\phi^2/2) \\ &\quad \times \text{Tr}_\sigma \exp \left[ \beta \sum_i [J \sigma_i \sigma_{i+1} + \mu_0 H \sigma_i] + \sqrt{p \sinh \beta I} \sum_i \sigma_i \phi \right].\end{aligned}\quad (24)$$

The transfer-matrix technique allows us to perform the summation over  $\sigma$  ( $\text{Tr}_\sigma$  in (24)) by replacing this summation by the product of  $N$   $2 \times 2$  matrices of the form

$$\begin{pmatrix} \exp[\beta J + \phi \sqrt{p \sinh \beta I} + \beta \mu_0 B] & \exp[-\beta J + \phi \sqrt{p \sinh \beta I} + \beta \mu_0 B] \\ \exp[-\beta J - \phi \sqrt{p \sinh \beta I} - \beta \mu_0 B] & \exp[\beta J - \phi \sqrt{p \sinh \beta I} - \beta \mu_0 B] \end{pmatrix} \quad (25)$$

which results in

$$\begin{aligned}\langle Z \rangle_I &= \frac{\sqrt{2}}{\sqrt{N\pi}} \int_{-\infty}^{\infty} d\phi \exp(-\phi^2 N/2) \left\{ \exp(\beta J) \cosh(\phi \sqrt{p \sinh \beta I} + \beta \mu_0 B) \right. \\ &\quad \left. + [\exp(2\beta J) \sinh^2(\phi \sqrt{p \sinh \beta I} + \beta \mu_0 B) + \exp(-2\beta J)]^{1/2} \right\}^N \\ &\equiv \int_{-\infty}^{\infty} d\phi [\exp(-\phi^2/2) \lambda(\phi)]^N.\end{aligned}\quad (26)$$

Since we will finally pass to the thermodynamical limit  $N \rightarrow \infty$ , the latter integral can be evaluated by the method of steepest descent. The saddle point(s) are defined by the condition  $\frac{d}{d\phi} [\exp(-\phi^2/2)\lambda(\phi)] = 0$  which has only one root  $\phi = 0$ , or an additional root defined by the following condition:

$$\phi = \frac{\sinh(\phi\sqrt{p \sinh \beta I} + \beta\mu_0 B)}{[\sinh^2(\phi\sqrt{p \sinh \beta I} + \beta\mu_0 B) + \exp(-4\beta J)]^{1/2}}. \quad (27)$$

In the absence of an external field,  $B = 0$ , equation (27) has a single solution  $\phi = 0$  for  $p \exp(\beta J) \sinh(\beta I) < 1$ , and an additional solution  $\phi = \phi(I, J, p) \neq 0$  for  $p \exp(\beta J) \sinh(\beta I) > 1$ . Clearly, the critical point  $T_c$  is defined by the following condition:

$$\exp\left(\frac{J}{\kappa T_c}\right) \sinh\left(\frac{I}{\kappa T_c}\right) = \frac{1}{p}. \quad (28)$$

There is no phase transition ( $T_c \rightarrow 0$ ) in the absence of random long-range interactions ( $p \rightarrow 0$ ), while in the absence of the short-range interactions ( $J = 0$ ), the critical point is defined by the following condition:

$$T_c = -\frac{J}{\kappa \ln[p/(1 + \sqrt{1 + p^2})]}. \quad (29)$$

For the equistrength case ( $I = J$ ) the critical point is equal to

$$T_c = -\frac{J}{\kappa \ln[2/(1 + \sqrt{1 + 8/p})]}. \quad (30)$$

Note that for  $p \ll 1$ , equations (29) and (30) have different limiting forms,

$$T_c|_{J=0} \simeq -\frac{J}{\kappa \ln(p/2)} \quad T_c|_{J=I} \simeq -\frac{2J}{\kappa \ln(p/2)}. \quad (31)$$

For the free energy  $\beta F = -\lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle Z \rangle_I$ , one obtains

$$\beta F = \int_{-\infty}^{\infty} d\phi \left\{ \frac{1}{2} \phi^2 p \sinh \beta J - \ln [\exp(\beta J) \cosh(\phi\sqrt{p \sinh \beta I} + \beta\mu_0 B) + [\exp(2\beta J) \sinh^2(\phi\sqrt{p \sinh \beta I} + \beta\mu_0 B) + \exp(-2\beta J)]^{1/2}] \right\}. \quad (32)$$

The free energy above the critical point is defined by the saddle point  $\phi = 0$ , which gives for  $B = 0$ , the well known result for the one-dimensional Ising system  $\beta F = -\ln(2 \cosh \beta J)$ , while below the critical point, one has to use the saddle point defined by (32).

The energy per site can be found from the free energy (32) using the well known thermodynamic relation  $E = \frac{\partial}{\partial \beta}(\beta F)$ , which gives for  $H = 0$ ,

$$E = -J \tanh \beta J \quad \text{for } T > T_c \quad (33)$$

and

$$E = J \left[ \frac{2\phi^2 \exp(-4\beta J)}{\sinh(\phi\sqrt{p \sinh \beta I}) [\sinh(\phi\sqrt{p \sinh \beta I}) + \phi \cosh(\phi\sqrt{p \sinh \beta I})]} - 1 - \frac{\phi^2}{2} \right] \quad \text{for } T < T_c. \quad (34)$$

As one can see from these equations, random long-range interactions induce the phase transition at  $T = T_c$ , while for high temperatures,  $T > T_c$ , the internal energy is defined only by the short-range interactions.

Using (27) and (28), one can see that the energy is continuous at the critical point and equal to  $-J \frac{\kappa T_c - 2J}{\kappa T_c + 2J}$ . On the other hand, the specific heat  $C = \frac{dE}{dT}$  has a finite jump at the critical point. From the free energy (32), one can also find the other critical indices and confirm that, as expected, the phase transition is of the second order with the mean-field critical indices.

Comparing the results of the two preceding sections, one concludes that the approximate method of section 3 missed an important result for the exact solution for the annealed case, namely, the existence of the lower limit for the concentration of long-range interactions below which no phase transition occurs.

#### 4. Quenched case

In the quenched case, one has to perform the average over distribution (2) of  $\ln Z$ , rather than of  $Z$ , as we did in previous sections for the annealed case. To this end, one has to use the so-called replica method [16], according to which one considers the  $n$  replicas  $\sigma_i^\alpha = \pm 1$  of each original  $\sigma_i$  and uses the following identity:

$$\langle \ln Z \rangle_J = \lim_{n \rightarrow 0} \frac{\langle Z^n \rangle_J - 1}{n}. \quad (35)$$

Introduction of different replicas in (20) immediately yields

$$\begin{aligned} \langle Z^n \rangle_J = \text{Tr}_\sigma \exp \left\{ \sum_\alpha \sum_{i=1}^N [\beta J \sigma_i^\alpha \sigma_{i+1}^\alpha + \beta \mu_0 B \sigma_i^\alpha] \right. \\ \left. + \sum_{i,j} \ln \left( 1 + \frac{p}{N} \left[ \exp \sum_\alpha (\beta I \sigma_i^\alpha \sigma_j^\alpha) - 1 \right] \right) \right\} \end{aligned} \quad (36)$$

where an additional summation over  $\alpha = 1, 2, \dots, n$  is added.

The following procedure [17] is similar to that performed in the previous section. Transforming the last exponent in (36) for each  $\alpha$  according to (21), substituting back in (36) and expanding the logarithm in a series in  $p/N$ , one finds instead of (22) the following expression:

$$\begin{aligned} \langle Z^n \rangle_J = \text{Tr}_\sigma \exp \sum_{\alpha, \alpha_1, \alpha_2, \dots} \left\{ \sum_{i=1}^N [\beta J \sigma_i^\alpha \sigma_{i+1}^\alpha + \beta \mu_0 B \sigma_i^\alpha] \right. \\ \left. + \frac{p}{N} \sum_{i,j} [(a_0 - 1) + a_1 \sigma_i^\alpha \sigma_j^\alpha + a_2 \sigma_i^\alpha \sigma_j^\alpha \sigma_i^{\alpha_1} \sigma_j^{\alpha_1} + a_3 \sigma_i^\alpha \sigma_j^\alpha \sigma_i^{\alpha_1} \sigma_j^{\alpha_1} \sigma_i^{\alpha_2} \sigma_j^{\alpha_2} + \dots] \right\} \end{aligned} \quad (37)$$

where

$$\lim_{n \rightarrow 0} a_m = \lim_{n \rightarrow 0} [\cosh^n(\beta J) \tanh^m(\beta J)] = \tanh^m(\beta J). \quad (38)$$

Equation (38) reduces to (21) when  $n = 1$  and  $m = 0, 1$ .

The next step is analogous to the transition from (22) to (27). Using the transformation (23) for each term in the summation over  $\alpha, \alpha_1, \alpha_2, \dots$  in (37), one obtains

$$\langle Z^n \rangle_J = \int d\mathbf{q} \exp[-Nf(\mathbf{q})] \quad (39)$$



where

$$f(q) = \frac{pa_1}{2} \sum_{\alpha} q_{\alpha}^2 + \frac{pa_2}{2} \sum_{\alpha, \alpha_1} q_{\alpha\alpha_1}^2 + \dots$$

$$- \ln \text{Tr}_{\sigma} \exp \left[ \sum_{\alpha, \alpha_1, \dots} [\beta J \sigma_{\alpha} \sigma_{\alpha+1} + \beta \mu_0 B \sigma_i^{\alpha} + pa_1 q_{\alpha} \sigma_{\alpha} + pa_2 q_{\alpha\alpha_1} \sigma_{\alpha} \sigma_{\alpha_1} + \dots] \right]$$
(40)

where the variables  $\sigma_{\alpha}$  represent  $n$  spins at the same site.

In contrast to (26), the last equation contains a series of order parameters  $q \equiv [q_{\alpha}, q_{\alpha\alpha_1}, \dots]$ . The integral in (39) can be found by the method of steepest descent analogously to (26) and (27). The average free energy (35) is then derived by minimization with respect to  $q_{\alpha}, q_{\alpha\alpha_1}, \dots$  and by taking the limit  $n \rightarrow 0$ . The general analysis is quite complicated, and it was performed either close to the phase transition point, where the situation becomes similar to the spin glasses and only two order parameters are important [17], or for low temperatures, where all order parameters are of the same amplitude [18, 19].

The details apart, an existence alone of phase transition in the quenched system with the random long-range interactions is important for our purposes. Another interesting question is the existence of a minimal average number  $p_1$  of these interactions per site needed for the appearance of a phase transition. It turns out [17, 19] that  $p_1 = 1$ . The latter result is connected with the theory of random graphs mentioned in section 1 [7], and, probably, does not depend on the dimensionality of the system, but is determined rather by the percolation: for  $p < 1$  and  $N \rightarrow \infty$  there is no macroscopic cluster of spins (no percolation), and hence no phase transition.

## 5. Conclusions

The Ising system with short-range and random long-range interactions has been considered by known methods in connection with recent intense interest in small-world networks. An analysis of a one-dimensional system confirms the main, intuitively clear result of Watts and Strogatz [1]: even a small amount of random long-range interactions drastically increases the connection between distant sites, which means, in the language of physics, the appearance of a phase transition.

It is common knowledge that the phase transition in a one-dimensional Ising system occurs for long-range infinitely small interactions between spins. The small-world approach shows that in order to obtain a phase transition, it is enough to have a small fraction (of the order of  $1/N$ ) of random interactions of a finite, distance-independent interactions of finite strength.

For physical systems, one has to distinguish between two types of long-range interactions. If the additional interaction between spins originated, say, from free electrons, then these interactions, like the spins themselves, are able to reach thermal equilibrium (annealed case). If, on the other hand, the additional interactions come, say, through impurities, then it is frozen (quenched case). We have considered these two cases separately and found that for both of them the appearance of phase transitions requires a minimal average number of long-range interactions per site,  $p_{\min}$ , with  $p_{\min} < 1$  for the annealed case, and  $p_{\min} \simeq 1$  for the quenched case. While the last result is probably correct for higher dimensions as well, our analysis was restricted to one dimension.

The whole situation becomes more interesting in two and three dimensions which are described by the same Hamiltonian (1) for which each of the indices  $i, j$  is defined by two or three numbers. Here, the phase transition already occurs in the ordered ( $p = 0$ ) system and

the appearance of a small disorder ( $p < 1$ ) will result in the change of the universality class of this transition, from Ising for  $p = 0$  to the mean field for  $p \neq 0$ . Again, the question arises as to the dependence of the critical indices on  $p$  in the transient regime. Work on this problem is in progress.

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